Study Guide for Exam 1

1. You are supposed to be able to determine the domain of a function, looking at the conditions for its expression to be well-defined. Some examples of the conditions are:

• What is inside of the square root must be non-negative (where there is no restriction on what is inside of the cubic root).

• The denominator must not be zero.

• What is inside of the logarithm must be strictly positive.

Example Problems

1.1. Find the domain of the following functions

(i) $y = f(x) = \sqrt{\ln x}$

(ii) $y = \ln\left[\frac{1}{\sqrt{t-3}} - \frac{1}{t}\right]$

(iii) $y = \frac{1}{x-1} + \frac{x}{\sqrt{x^2-9}}$

2. Having the information on the range of a given rotation angle θ and knowing the value of a trigonometric function, you are supposed to be able to determine the values of the other trigonometric functions.

Example Problems

2.1. We have the information

$$\sin\theta = \frac{3}{5} \text{ and } \frac{\pi}{2} < \theta < \pi.$$

Determine the values of $\cos\theta, \tan\theta, \csc\theta, \sec\theta, \cot\theta$.

2.2. We have the information

$$\tan\theta = -\frac{5}{12}, \cos\theta < 0, \sin\theta > 0.$$

Determine the values of $\sin\theta, \cos\theta, \csc\theta, \sec\theta, \cot\theta$.

3. You are supposed to be able to solve the equations involving the trigonometric functions and find solutions on the given interval, using the basic formulas of the trigonometric functions (e.g., double angle formula for sine and cosine, $\sin^2 x + \cos^2 x = 1$, etc.).

Example Problems

3.1. Find the values of x on the interval $[0, 2\pi]$ which satisfy the equation

$$\cos x = \cos(2x).$$

3.2. Find the values of x on the interval $[0, 2\pi]$ which satisfy the equation

$$3\cot x = 2\sin(2x).$$

4. Starting from the graph of a given function, you are supposed to know how to shift vertically/horizontally and/or reflect with respect to a certain line, to draw the graph of a new function, depending on the formula for the new function. Conversely, you are supposed to know how to derive the formula for the new function, once you are given the information on its graph through shifts, reflection compared to the original graph.

Example Problems

4.1. Find the formula for a function

(i) whose graph is symmetric with respect to a line $y = -3$ to the graph of $y = e^x$.

(ii) whose graph is symmetric with respect to a line $x = 1$ to the graph of $y = |x - 5|$.

4.2. We consider the function $f(x) = 2e^x - 3$.

Firstly we reflect the graph of $y = f(x)$ with respect to $x = 3$.

Secondly we reflect the resulting gragh with respect to $y = -1$.

Find the equation representing the function associated with the final graph and its range.

5. Given a function which is one-to-one, you are supposed to be able to find the formula of the inverse function, its domain and range, and draw the graph of the inverse function.

Example Problems

5.1. Find the formula, and state the domain and range of the inverse of the function

$$y = \frac{1 - e^{-x}}{1 + e^{-x}}.$$

5.2.

(i) Consider the function $y = f(x) = x^2 - 1$ over the interval $(-2, 0]$. Find the formula, and state the domain and range of its inverse.

(ii) Consider the function $y = f(x) = x^2 - 1$ over the interval $[1, 3)$. Find the formula, and state the domain and range of its inverse.

5.3. Consider the function $y = \dfrac{3x + 1}{-5x + 3}$. Find the formula representing the function whose graph is symmetric, with respect to the line $y = x$, to the graph of the given function.

6. You are supposed to be able to solve the equations involving the exponential and logarithmic functions.

Example Problems

6.1. Look at the problems 51, 52, 53, 54 on Page 67 of the textbook.

7. You are supposed to be able to compute the right/left hand side limit, understanding its proper meaning. You are also supposed to determine the exact value of the limit who has an indeterminate form on the surface.

Example Problems

7.1. Compute the following limits:

(i) $\lim_{x\to 5^-} \dfrac{x^2-3x-5}{|x-5|}$

(ii) $\lim_{x\to(\pi/2)^-} \tan x$

(iii) $\lim_{x\to(\pi/2)^+} e^{\tan x}$

(iv) $\lim_{x\to 0} \left(\dfrac{5}{x^2-x} + \dfrac{5}{x}\right)$

(v) $\lim_{x\to\infty} \left(\sqrt{x^2+3x+1} - x\right)$

(vi) $\lim_{x\to 0} \dfrac{|3x-4|-|5x+4|}{x}$

(vii) $\lim_{x\to -3} \dfrac{2|x|-6}{x+3}$

8. You are supposed to understand the meaning of the Intermediate Value Theorem, and to be able use it to show that a certain equation has a root in a specified interval.

Example Problems

8.1. Using the I.V. Th., determine on which of the following intervals

$$(0,1), (1,2), (2,3), (3,4)$$

the equation $x^3-3x=5$ has a root.

8.2. Prove that the following equation $\sin x = x^3+1$ has at least one real root.

9. You are supposed to understand the meaning of the defining formula of the derivative, and being able to determine the values of the related limits.

Example Problems

9.1. Suppose we have a function $f(x)$ with $f'(2)=5$.

Determine the following values:

(i) $g'(1)$ where $g(x)=f(2x)$.

(ii) $\lim_{h\to 0} \dfrac{f(2+4h)-f(2)}{3h}$.

(iii) $\lim_{h\to 0} \dfrac{f(2+4h)-f(2+5h)}{9h}$.

9.2. Let $f(x)$ be the function defined as follows:

$$f(x) = \begin{cases} -(x-1)^2 + 3 & \text{if} \quad x < 0 \\ 2e^x & \text{if} \quad x \geq 0. \end{cases}.$$

Does $f'(0)$ exist ? If it exists, compute the value of $f'(0)$.

10. When a function is defined piecewise and depending on some variables, you are supposed to know how to determine those variables so that the function becomes continuous entirely over the specified interval (e.g. everywhere).

Example Problems

10.1. Find the values of a and b so that the function

$$f(x) = \begin{cases} x^2 - a & \text{if} \quad x \leq 1 \\ \dfrac{3x^2 + 12x - b}{x^2 + 2x - 3} & \text{if} \quad x > 1 \end{cases}$$

is continuous on $(-\infty, \infty)$.

10.2. Consider the following function

$$f(x) = \begin{cases} 3x - 2c & \text{if} \quad x \leq c \\ 5x^2 - 4 & \text{if} \quad x > c. \end{cases}$$

Determine all the value of c so that f is continuous everywhere.

11. You are supposed to be able to find the horizontal/vertical asymptote(s) of a given function.

Example Problems

11.1. Find the horizontal/vertical asymptote(s) of the following functions

(i) $y = f(x) = \dfrac{x^3 + 4x^2 + x - 6}{x(x^2 - 1)}$.

(ii) $y = f(x) = \dfrac{x^2 - x}{x^2 - 4x + 3}$.

(iii) $y = f(x) = \dfrac{3e^x}{e^x - 1}$.

12. You are supposed to be able to compute the limits, using the Squeeze Theorem and some basic limits such as $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta}$.

Example Problems

12.1. Compute the following limits:

(i) $\lim_{x \to 0} \dfrac{\sin(3x)}{5x}$

(ii) $\lim_{x\to\infty} \dfrac{\sin(3x)}{5x}$
(iii) $\lim_{x\to 0} \dfrac{\sin(1/x)}{1/x}$
(iv) $\lim_{x\to 0} \dfrac{\sin(4x)^2}{2x^2 \cos x}$
(v) $\lim_{x\to -\infty} \cos(3x)e^x$

13. You are supposed to be able to compute the derivative of a function, and understand that its value represents the slope of the tangent line to the graph of the function.

Example Problems

13.1. Find the equation of the line that is tangent to the curve $y = x^3$ and is
(i) parallel to the line $y = 3x$,
(ii) perpendicular to the line $y = 3x$.
13.2. Find the equation of the line(s) which is tangent to the parabola $y = x^2$ and passes thorugh the point $(1, -3)$.

14. You are supposed to be able to compute the derivative using the power, product, and quotient rules. You are also supposed to be able to compute the derivatives of the trigonometric functions.

Example Problems

14.1. Compute the derivative of the following function.
(i) $f(x) = x^2 e^x$.
(ii) $f(x) = \dfrac{1+x}{1-x}$.
14.2. Given $f(x) = e^x g(x)$ and $g(0) = 3, g'(0) = 5, g''(0) = 7$, compute $f''(0)$.

Challenge Problem: Consider the function described below:

$$g(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } \ x \neq 0 \\ 0 & \text{if } \ x = 0 \end{cases}$$

Choose the right statement about the continuity and differentiability of the function $y = g(x)$ at 0.

A. The function g is continuous at 0 and differentiable at 0.
B. The function g is continuous at 0 but not differentiable at 0.
C. The function g is not continuous at 0 but differentiable at 0.
D. The function g is not continuous at 0 and not differentiable at 0.
E. The above description is not sufficient to judge the continuity or the differentiability of the function g.